Name of the Course - B.Sc(H) Physics

Semester - IV

Paper Name - Mathematical Physics III

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Instructions to the Students:-

① Dear Students I am sharing the last three topics from Laplace transform.

② The first one is the Solution of ordinary differential equation by Laplace transform. For any eqn to be solved you have to take Laplace transform of the given eqn both sides and then you have to take inverse Laplace transform as we have Sol^n differential eq^n in fourier tranform. When taking inverse Laplace transform you have to remember all the Laplace transform you have to remember all the inverse Laplace tranform properties to solve it.

3) The second topic is Solution of Simultaneous differential eq^n of first order, it is also called Coupled differential eq^n. In this type of differential eq^n you have two eq^ns given. You have to take Laplace transform

both sides of a given differential eqn. You will get two eqn in term of inverse Laplace transform and you have to solve the given differential eqn by the method of Addition and Subtraction and then final take inverse tranform both sides then you will get the final solution.

3) The last topic is "the sol of heat flow along semi-infinite bar using Laplace transform". I have collected this e-notes from Schaum's Series book and I didn't find this topic anywhere. Here you have to understand the error function and its Laplace transform. You all first study this topic and then if you will find any difficulty then I will help you to understand.

This topic comes as long answer type of 15 marks.

Application of Laplace transform in Coupled / Simultaneous differential equation

Q:→

$$\frac{dx}{dt} + x + y = 0$$

$$\frac{dy}{dt} + 4x + y = 0 \quad \text{, given } x(0) = y(0) = 1$$

Solⁿ:→ Here, we have

$$\frac{dx}{dt} + x + y = 0 \qquad \text{—— (1)}$$

$$\frac{dy}{dt} + 4x + y = 0 \qquad \text{—— (2)}$$

Taking Laplace transform of (1) both Sides we get

$$L\left\{\frac{dx}{dt}\right\} + L\{x\} + L\{y\} = 0$$

$$\Rightarrow L\{x'\} + L\{x\} + L\{y\} = 0$$

$$S\bar{x} - x(0) + \bar{x} + \bar{y} = 0 \qquad \text{—— (3)} \quad (\text{where, } \bar{x} = L(x) \text{ and } \bar{y} = L(y))$$

Again taking Laplace transform of (2) on both Sides we get,

$$S\bar{y} - y(0) + 4\bar{x} + \bar{y} = 0$$

$$\rightarrow (S+1)\bar{y} + 4\bar{x} = 1 \qquad \text{—— (4)} \quad [\because y(0) = 1]$$

Multiplying (3) by (S+1) and Subtracting from (4) we get

$(S+1)\bar{y} + 4\bar{x} = 1$

$\bar{y}(S+1) + (S+1)\bar{x} + S(S+1)\bar{x} = S+1$

$$S\bar{x}(S+1) - x(0)(S+1) + \bar{x}(S+1) + \bar{y}(S+1) = 0$$

$$S(S+1)\bar{x} - (S+1) + \bar{x}(S+1) + \bar{y}(S+1) = 0$$

$$\bar{x}\{S(S+1) + (S+1)\} + \bar{y}(S+1) - (S+1) = 0$$

$$\bar{x}(S+1)(S+1) + \bar{y}(S+1) - (S+1) = 0$$

$$\bar{x}(S+1)^2 + \bar{y}(S+1) - (S+1) = 0 \qquad \text{—— (3a)}$$

Now Subtracting this (3a) from (4) we get

$$(S+1)\bar{y} + 4\bar{x} = 1$$
$$-\;(S+1)\bar{y} + (S+1)^2\bar{x} = (S+1)$$

$$4\bar{x} - (S+1)^2\bar{x} = \cancel{1} - S - \cancel{1}$$
$$4\bar{x} - (S+1)^2\bar{x} = -S$$
$$\therefore (S+1)^2\bar{x} - 4\bar{x} = S$$
$$\therefore \bar{x} = \frac{S}{(S+1)^2-4} = \frac{S+1}{(S+1)^2-4} - \frac{1}{(S+1)^2-4} \quad \text{— (5)}$$

Taking inverse transform of (5) we get,

$$L^{-1}\{\bar{x}\} = L^{-1}\left\{\frac{S+1}{(S+1)^2-4}\right\} - L^{-1}\left\{\frac{1}{(S+1)^2-4}\right\}.$$

$$\left[x = e^{-t}\cosh 2t - \frac{1}{2}e^{-t}\cdot\sinh 2t\right] \quad \text{— (6)}$$

Again multiplying (3) by 4 and (4) by (S+1) then Subtract we get.

$$4\cdot S\bar{x} - 4x(0) + 4\bar{x} + 4\bar{y} = 0 \quad \text{— (3a)}$$
$$(S+1)^2\bar{y} + 4(S+1)\bar{x} = (S+1) \quad \text{— (4a)}$$

$$(S+1)^2\bar{y} + 4(S+1)\bar{x} = (S+1)$$
$$4\bar{x} + 4\bar{y} + 8\bar{x} = 4$$
$$4\bar{y} - (S+1)^2\bar{y} = S+1-4$$
$$4\bar{y} - (S+1)^2\bar{y} = S-3$$
$$\bar{y} = \frac{S-3}{(S+1)^2-4}$$

Subtracting (4a) from (3a)

$$4\bar{y} - (S+1)^2\bar{y} = -S+3$$

$$S - 3 = \{4 + (S+1)^2\}\bar{y} \Rightarrow \bar{y} = \frac{S-3}{(S+1)^2-4}$$

Taking inverse transform,

$$L\{\bar{y}\} = L^{-1}\left\{\frac{S+1}{(S+1)^2-4}\right\} - 4L^{-1}\left\{\frac{1}{(S+1)^2-4}\right\} \qquad \bar{y} = \frac{S-3}{\{4-(S+1)^2\}} \qquad \cancel{\frac{S-3}{(S+1)^2-4}}$$

$$= \left[e^{-t}\cdot\cosh 2t - \frac{4}{2}e^{-t}\sinh 2t\right] \qquad \bar{y} = \frac{S+1}{(S+1)^2-4} - \frac{4}{(S+1)^2-4} \qquad \frac{S+1-1-3}{(S+1)^2-4}$$

$$= \left[e^{-t}\cosh 2t - 2e^{-t}\sinh 2t\right]$$

Application of Laplace transform in Solving ordinary differential eqn:-

① Q - Solve by Laplace method,

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dy} + y = e^t \quad \text{with } y(0) = 2,\ y'(0) = -1$$

Ist Step → Taking Laplace transform both Sides.

$$L\left\{\frac{d^2y}{dt^2}\right\} - 2L\left\{\frac{dy}{dt}\right\} + L\{y\} = L\{e^t\}$$

$L\{y'(t)\} = s\bar{y}(s) - y(0)$
$L\{y''(t)\} = s^2\bar{y}(s) - sy(0) - y'(0)$

putting value of $L\{y'(t)\}$ and $L\{y''(t)\}$ we get

$$s^2\bar{y}(s) - s(\underset{2}{y(0)}) - \underset{-1}{y'(0)} - 2[s\bar{y}(s) - \underset{2}{y(0)}] - \bar{y}(s) = \frac{1}{s-1}$$

→ we have to find $y(t)$

when $\bar{y}(s) = L\{y(t)\}$

$$\Rightarrow s^2\bar{y}(s) - 2s - (-1) - 2[s\bar{y}(s) - 2] + \bar{y}(s) = \frac{1}{s-1}$$

$$s^2\bar{y}(s) - 2s + 1 - 2s\bar{y}(s) + 4 + \bar{y}(s) = \frac{1}{s-1}$$

Taking $\bar{y}(s)$ Common

$$(s^2 - 2s + 1)\bar{y}(s) - 2s + 1 + 4 = \frac{1}{s-1}$$

$$(s-1)^2\bar{y}(s) = \frac{1}{s-1} + 2s - 5 = \frac{2s^2 - 7s + 5 + 1}{s-1} = \frac{2s^2 - 7s + 6}{}$$

$$= \frac{1 + (2s-5)(s-1)}{s-1} = \frac{1 + 2s^2 - 5s + 5 - 2s}{(s-1)} = \frac{2s^2 - 7s + 6}{(s-1)}$$

$$\Rightarrow \bar{y}(s) = \frac{2s^2 - 7s + 6}{(s-1)^3}$$

$$L\{y(t)\} = \frac{2s^2 - 7s + 6}{(s-1)^3}$$

Taking inverse Laplace transform both Sides,

$$y(t) = L^{-1}\left\{\frac{2s^2 - 7s + 6}{(s-1)^3}\right\} \quad \text{— ①}$$

Using partial fraction method,

$$\frac{2s^2 - 7s + 6}{(s-1)^3} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3}$$

$$2s^2 - 7s + 6 = A(s-1)^2 + B(s-1) + C$$

$$= A(s^2 - 2s + 1) + B(s-1) + C =$$

$$2s^2 - 7s + 6 = As^2 + s(-2A + B) + (A + C - B)$$

$$A + C - B = 6$$
$$2 + C + 3 = 6$$
$$C = 1.$$

Equating the Coefficient of s^2, s, s^0 we get.

$$\boxed{A = 2}, \quad -2A + B = -7 \Rightarrow \boxed{B = -3}, \quad \cancel{A + C = 6} \quad A + C - B = 6$$

$$\therefore -2 \times 2 + B = -7$$
$$-4 + B = -7 \Rightarrow B = -3$$

$$\boxed{C = 1}$$

Putting the value of A, B, C in Eq^n (1)

$$Y(t) = L^{-1}\left\{\frac{2}{s-1} - \frac{3}{(s-1)^2} + \frac{1}{(s-1)^3}\right\}$$

$$= L^{-1}\left\{\frac{2}{s-1}\right\} - L^{-1}\left\{\frac{3}{(s-1)^2}\right\} + L^{-1}\left\{\frac{1}{(s-1)^3}\right\}$$

$$Y(t) = 2e^t - 3e^t L^{-1}\left\{\frac{1}{s^2}\right\} + e^t L^{-1}\left(\frac{1}{s^3}\right\}$$

$$= 2e^t - 3e^t \cdot t + e^t \cdot \frac{t^2}{\lfloor 2}$$

$$\left[Y(t) = 2e^t - 3e^t \cdot t + t^2 e^t\right]$$

Topic 3: solution of heat equation taken from schaum's sereis laplace transform book

differential equation. In such case the required solution is obtained ... The process is sometimes referred to as *iterated Laplace transformation*. A similar technique can be applied to three (or higher) dimensional problems. Boundary-value problems can sometimes also be solved by using both Fourier and Laplace transforms [see Prob. 14].

Solved Problems

HEAT CONDUCTION

1. A semi-infinite solid $x > 0$ [see Fig. 8-5] is initially at temperature zero. At time $t = 0$, a constant temperature $U_0 > 0$ is applied and maintained at the face $x = 0$. Find the temperature at any point of the solid at any later time $t > 0$.

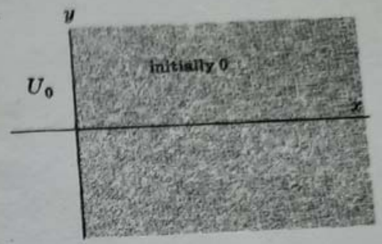

Fig. 8-5

The boundary-value problem for the determination of the temperature $U(x, t)$ at any point x and any time t is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} \qquad x>0,\ t>0$$

$$U(x,0) = 0, \qquad U(0,t) = U_0, \qquad |U(x,t)| < M$$

where the last condition expresses the requirement that the temperature is bounded for all x and t.

Taking Laplace transforms, we find

$$su - U(x,0) = k\frac{d^2u}{dx^2} \quad \text{or} \quad \frac{d^2u}{dx^2} - \frac{s}{k}u = 0 \tag{1}$$

where

$$u(0,s) = \mathcal{L}\{U(0,t)\} = \frac{U_0}{s} \tag{2}$$

and $u = u(x,s)$ is required to be bounded.

Solving (1), we find

$$u(x,s) = c_1 e^{\sqrt{s/k}\,x} + c_2 e^{-\sqrt{s/k}\,x}$$

Then we choose $c_1 = 0$ so that u is bounded as $x \to \infty$, and we have

$$u(x,s) = c_2 e^{-\sqrt{s/k}\,x} \tag{3}$$

From (2) we have $c_2 = U_0/s$, so that

$$u(x,s) = \frac{U_0}{s} e^{-\sqrt{s/k}\,x}$$

Hence by Problem 9, Page 207, and Problem 10, Page 209, we find

$$U(x,t) = U_0 \operatorname{erfc}(x/2\sqrt{kt}) = U_0\left\{1 - \frac{2}{\sqrt{\pi}}\int_0^{x/2\sqrt{kt}} e^{-u^2}\,du\right\}$$

2. Work Problem 1 if at $t=0$ the temperature applied is given by $G(t)$, $t>0$.

The boundary-value problem in this case is the same as in the preceding problem except that the boundary condition $U(0,t) = U_0$ is replaced by $U(0,t) = G(t)$. Then if the Laplace transform of $G(t)$ is $g(s)$, we find from (3) of Problem 1 that $c_2 = g(s)$ and so

$$u(x,s) = g(s)\, e^{-\sqrt{s/k}\,x}$$

Now, residue at $s = -1$ is

$$\lim_{s\to -1} \frac{1}{2!} \frac{d^2}{ds^2}\left[(s+1)^3 \frac{se^{st}}{(s+1)^3(s-1)^2}\right] \;=\; \lim_{s\to -1} \frac{1}{2} \frac{d^2}{ds^2}\left[\frac{se^{st}}{(s-1)^2}\right] \;=\; \frac{1}{16} e^{-t}(1-2t^2)$$

and residue at $s = 1$ is

$$\lim_{s\to 1} \frac{1}{1!} \frac{d}{ds}\left[(s-1)^2 \frac{se^{st}}{(s+1)^3(s-1)^2}\right] \;=\; \lim_{s\to 1} \frac{d}{ds}\left[\frac{se^{st}}{(s+1)^3}\right] \;=\; \frac{1}{16} e^{t}(2t-1)$$

Then $$\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^3(s-1)^2}\right\} \;=\; \sum \text{residues} \;=\; \frac{1}{16}e^{-t}(1-2t^2) + \frac{1}{16}e^{t}(2t-1)$$

8. Evaluate $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+1)^2}\right\}$.

We have $$\frac{1}{(s^2+1)^2} \;=\; \frac{1}{[(s+i)(s-i)]^2} \;=\; \frac{1}{(s+i)^2(s-i)^2}$$

The required inverse is the sum of the residues of

$$\frac{e^{st}}{(s+i)^2(s-i)^2}$$

at the poles $s = i$ and $s = -i$ which are of order two each.

Now, residue at $s = i$ is

$$\lim_{s\to i} \frac{d}{ds}\left[(s-i)^2 \frac{e^{st}}{(s+i)^2(s-i)^2}\right] \;=\; -\frac{1}{4}te^{it} - \frac{1}{4}ie^{it}$$

and residue at $s = -i$ is

$$\lim_{s\to -i} \frac{d}{ds}\left[(s+i)^2 \frac{e^{st}}{(s+i)^2(s-i)^2}\right] \;=\; -\frac{1}{4}te^{-it} + \frac{1}{4}ie^{-it}$$

which can also be obtained from the residue at $s = i$ by replacing i by $-i$. Then

$$\begin{aligned} \sum \text{residues} &= -\frac{1}{4}t(e^{it}+e^{-it}) - \frac{1}{4}i(e^{it}-e^{-it}) \\ &= -\frac{1}{2}t\cos t + \frac{1}{2}\sin t \;=\; \frac{1}{2}(\sin t - t\cos t) \end{aligned}$$

Compare with Problem 18, Page 54.

Page-207

What is Complex inversion formula

INVERSE LAPLACE TRANSFORMS OF FUNCTIONS WITH BRANCH POINTS

9. Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-a\sqrt{s}}}{s}\right\}$ by use of the complex inversion formula.

By the complex inversion formula, the required inverse Laplace transform is given by

$$F(t) \;=\; \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st - a\sqrt{s}}}{s}\, ds \qquad (1)$$

Since $s = 0$ is a branch point of the integrand, we consider

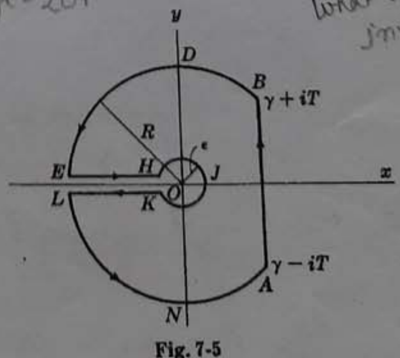

Fig. 7-5

$$\begin{aligned}\frac{1}{2\pi i}\oint_C \frac{e^{st-a\sqrt{s}}}{s}\,ds \;=\; & \frac{1}{2\pi i}\int_{AB}\frac{e^{st-a\sqrt{s}}}{s}\,ds + \frac{1}{2\pi i}\int_{BDE}\frac{e^{st-a\sqrt{s}}}{s}\,ds \\ & + \frac{1}{2\pi i}\int_{EH}\frac{e^{st-a\sqrt{s}}}{s}\,ds + \frac{1}{2\pi i}\int_{HJK}\frac{e^{st-a\sqrt{s}}}{s}\,ds \\ & + \frac{1}{2\pi i}\int_{KL}\frac{e^{st-a\sqrt{s}}}{s}\,ds + \frac{1}{2\pi i}\int_{LNA}\frac{e^{st-a\sqrt{s}}}{s}\,ds\end{aligned}$$

where C is the contour of Fig. 7-5 consisting of the line AB $(s=\gamma)$, the arcs BDE and LNA of a circle of radius R and center at origin O, and the arc HJK of a circle of radius ϵ with center at O.

Since the only singularity $s=0$ of the integrand is not inside C, the integral on the left is zero by Cauchy's theorem. Also, the integrand satisfies the condition of Problem 2 [see Problem 61] so that on taking the limit as $R\to\infty$ the integrals along BDE and LNA approach zero. It follows that

$$\begin{aligned}F(t) \;=\; & \lim_{\substack{R\to\infty\\ \epsilon\to 0}} \frac{1}{2\pi i}\int_{AB}\frac{e^{st-a\sqrt{s}}}{s}\,ds \;=\; \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{st-a\sqrt{s}}}{s}\,ds \\ =\; & -\lim_{\substack{R\to\infty\\ \epsilon\to 0}} \frac{1}{2\pi i}\left\{\int_{EH}\frac{e^{st-a\sqrt{s}}}{s}\,ds + \int_{HJK}\frac{e^{st-a\sqrt{s}}}{s}\,ds + \int_{KL}\frac{e^{st-a\sqrt{s}}}{s}\,ds\right\} \qquad (2)\end{aligned}$$

Along EH, $s = xe^{\pi i}$, $\sqrt{s} = \sqrt{x}\,e^{\pi i/2} = i\sqrt{x}$ and as s goes from $-R$ to $-\epsilon$, x goes from R to ϵ. Hence we have

$$\int_{EH}\frac{e^{st-a\sqrt{s}}}{s}\,ds \;=\; \int_{-R}^{-\epsilon}\frac{e^{st-a\sqrt{s}}}{s}\,ds \;=\; \int_R^{\epsilon}\frac{e^{-xt-ai\sqrt{x}}}{x}\,dx$$

Similarly, along KL, $s = xe^{-\pi i}$, $\sqrt{s} = \sqrt{x}\,e^{-\pi i/2} = -i\sqrt{x}$ and as s goes from $-\epsilon$ to $-R$, x goes from ϵ to R. Then

$$\int_{KL}\frac{e^{st-a\sqrt{s}}}{s}\,ds \;=\; \int_{-\epsilon}^{-R}\frac{e^{st-a\sqrt{s}}}{s}\,ds \;=\; \int_{\epsilon}^{R}\frac{e^{-xt+ai\sqrt{x}}}{x}\,dx$$

Along HJK, $s = \epsilon e^{i\theta}$ and we have

$$\begin{aligned}\int_{HJK}\frac{e^{st-a\sqrt{s}}}{s}\,ds \;&=\; \int_{\pi}^{-\pi}\frac{e^{\epsilon e^{i\theta}t-a\sqrt{\epsilon}\,e^{i\theta/2}}}{\epsilon e^{i\theta}}\,i\epsilon e^{i\theta}\,d\theta \\ &=\; i\int_{\pi}^{-\pi} e^{\epsilon e^{i\theta}t-a\sqrt{\epsilon}\,e^{i\theta/2}}\,d\theta\end{aligned}$$

Thus (2) becomes

$$\begin{aligned}F(t) \;&=\; -\lim_{\substack{R\to\infty\\ \epsilon\to 0}} \frac{1}{2\pi i}\left\{\int_R^{\epsilon}\frac{e^{-xt-ai\sqrt{x}}}{x}\,dx + \int_{\epsilon}^{R}\frac{e^{-xt+ai\sqrt{x}}}{x}\,dx + i\int_{\pi}^{-\pi}e^{\epsilon e^{i\theta}t-a\sqrt{\epsilon}\,e^{i\theta/2}}\,d\theta\right\} \\ &=\; -\lim_{\substack{R\to\infty\\ \epsilon\to 0}} \frac{1}{2\pi i}\left\{\int_{\epsilon}^{R}\frac{e^{-xt}(e^{ai\sqrt{x}}-e^{-ai\sqrt{x}})}{x}\,dx + i\int_{\pi}^{-\pi}e^{\epsilon e^{i\theta}t-a\sqrt{\epsilon}\,e^{i\theta/2}}\,d\theta\right\} \\ &=\; -\lim_{\substack{R\to\infty\\ \epsilon\to 0}} \frac{1}{2\pi i}\left\{2i\int_{\epsilon}^{R}\frac{e^{-xt}\sin a\sqrt{x}}{x}\,dx + i\int_{\pi}^{-\pi}e^{\epsilon e^{i\theta}t-a\sqrt{\epsilon}\,e^{i\theta/2}}\,d\theta\right\}\end{aligned}$$

Since the limit can be taken underneath the integral sign, we have

$$\lim_{\epsilon\to 0}\int_{\pi}^{-\pi}e^{\epsilon e^{i\theta}t-a\sqrt{\epsilon}\,e^{i\theta/2}}\,d\theta \;=\; \int_{\pi}^{-\pi}1\,d\theta \;=\; -2\pi$$

and so we find

$$F(t) \;=\; 1-\frac{1}{\pi}\int_0^{\infty}\frac{e^{-xt}\sin a\sqrt{x}}{x}\,dx \qquad (3)$$

This can be written (see Problem 10) as

$$F(t) \;=\; 1-\operatorname{erf}(a/2\sqrt{t}) \;=\; \operatorname{erfc}(a/2\sqrt{t}) \qquad (4)$$

10. Prove that $\frac{1}{\pi}\int_0^\infty \frac{e^{-xt}\sin a\sqrt{x}}{x}\,dx = \operatorname{erf}(a/2\sqrt{t})$ and thus establish the final result (4) of Problem 9.

Letting $x = u^2$, the required integral becomes

$$I = \frac{2}{\pi}\int_0^\infty \frac{e^{-u^2t}\sin au}{u}\,du$$

Then differentiating with respect to a and using Problem 183, Page 41,

$$\frac{\partial I}{\partial a} = \frac{2}{\pi}\int_0^\infty e^{-u^2t}\cos au\,du = \frac{2}{\pi}\left(\frac{\sqrt{\pi}}{2\sqrt{t}}e^{-a^2/4t}\right) = \frac{1}{\sqrt{\pi t}}e^{-a^2/4t}$$

Hence, using the fact that $I = 0$ when $a = 0$,

$$I = \int_0^a \frac{1}{\sqrt{\pi t}}e^{-p^2/4t}\,dp = \frac{2}{\sqrt{\pi}}\int_0^{a/2\sqrt{t}} e^{-u^2}\,du = \operatorname{erf}(a/2\sqrt{t})$$

and the required result is established.

11. Find $\mathcal{L}^{-1}\{e^{-a\sqrt{s}}\}$.

If $\mathcal{L}\{F(t)\} = f(s)$, then we have $\mathcal{L}\{F'(t)\} = s\,f(s) - F(0) = s\,f(s)$ if $F(0) = 0$. Thus if $\mathcal{L}^{-1}\{f(s)\} = F(t)$ and $F(0) = 0$, then $\mathcal{L}^{-1}\{s\,f(s)\} = F'(t)$.

By Problems 9 and 10, we have

$$F(t) = \operatorname{erfc}(a/2\sqrt{t}) = 1 - \frac{2}{\sqrt{\pi}}\int_0^{a/2\sqrt{t}} e^{-u^2}\,du$$

so that $F(0) = 0$ and
$$f(s) = \mathcal{L}\{F(t)\} = \frac{e^{-a\sqrt{s}}}{s}$$

Then it follows that

$$\mathcal{L}^{-1}\{e^{-a\sqrt{s}}\} = F'(t) = \frac{d}{dt}\left\{1 - \frac{2}{\sqrt{\pi}}\int_0^{a/2\sqrt{t}} e^{-u^2}\,du\right\}$$

$$= \frac{a}{2\sqrt{\pi}}\,t^{-3/2}\,e^{-a^2/4t}$$

INVERSE LAPLACE TRANSFORMS OF FUNCTIONS WITH INFINITELY MANY SINGULARITIES

12. Find all the singularities of $f(s) = \dfrac{\cosh x\sqrt{s}}{s\cosh\sqrt{s}}$ where $0 < x < 1$.

Because of the presence of $\sqrt{s}$, it would appear that $s = 0$ is a branch point. That this is not so, however, can be seen by noting that

$$f(s) = \frac{\cosh x\sqrt{s}}{s\cosh\sqrt{s}} = \frac{1 + (x\sqrt{s})^2/2! + (x\sqrt{s})^4/4! + \cdots}{s\{1 + (\sqrt{s})^2/2! + (\sqrt{s})^4/4! + \cdots\}}$$

$$= \frac{1 + x^2s/2! + x^4s^2/4! + \cdots}{s\{1 + s/2! + s^2/4! + \cdots\}}$$

from which it is evident that there is no branch point at $s = 0$. However, there is a simple pole at $s = 0$.

The function $f(s)$ also has infinitely many poles given by the roots of the equation

$$\cosh\sqrt{s} = \frac{e^{\sqrt{s}} + e^{-\sqrt{s}}}{2} = 0$$