COURSE : B . Sc (Hons) PHYSICS

CLASS : SEMESTER-6, SECTION-B

PAPER : ADVANCED MATHEMATICAL PHYSICS-II

TEACHER : SEEMA TRAMA

TOPIC : INTRODUCTION TO GROUPS

The following are the REFERENCES for the 'GROUP THEORY' section of the syllabus:

M . Hammermesh : Group Theory and its Applications to Physical Problems.(MH)

A. W. Joshi : Elements of Group Theory for Physicists .(AWJ)

A.K. Vasishtha & A.R. Vasishtha: Modern Algebra(AKV)

READINGS FOR THIS TOPIC :

Chapter -1(Pages 1-30 MH)

Chapter -1(Pages 1-29AWJ)

Chapter -2(Pages 1-34AKV)

GROUP: A set of distinct elements G = {E , a , b ,c….} together with a law of composition (for example: addition , multiplication ,matrix multiplication…..etc.) denoted (in general) by "$*$" is said to be a group if the following properties hold:

1) **CLOSURE : The composition of any of the two elements say, a and b of G results in an element which belongs to the set G. This can be mathematically represented as :**

   If $a , b \in G$

   $a * b = c \in G \quad \forall\, a , b \in G$

   The above property is called closure property and the set is said to be "closed" with respect to the composition "*".

2) **ASSOCIATIVITY : The said law of composition "*" is said to be associative if for any elements a, b, c $\in$ G:**

   $a * (b * c) = (a * b) * c \quad \forall\, a , b , c \in G$

3) <u>EXISTENCE OF A UNIQUE IDENTITY ELEMENT (FOR THE WHOLE GROUP)</u> : There exists a unique element called the " identity " and denoted by "E" or "e" or "I" ϵ G such that

$E*a = a*E = a \quad \forall a \epsilon G.$

4) <u>EXISTENCE OF AN INVERSE (FOR EVERY ELEMENT OF THE GROUP)</u> : Each element of G possesses a unique inverse . For an element a ϵ G ther exists a unique element b ϵ G such that:

$a * b = b * a = e .$

The element "b" is called the inverse of "a" or

$b = a^{-1}$

Note: a^{-1} must also be an element of G.

If all the above properties hold for a set , it is said to be a group wrt the binary operation "*" or we can say that the algebraic structure (G,*) is a group if the above postulates hold for the binary operation "*".

<u>ABELIAN GROUP</u> : A group G is said to be "ABELIAN" if ,in addition to the above four postulates, the following postulate is also obeyed:

$a * b = b * a \quad \forall a , b \epsilon G$.

That is ,a group all of whose elements commute with each other is said to be abelian.

<u>ORDER OF A GROUP</u> : The number of elements in a group is called its order.

If a group possesses an infinite number of elements , then it is said to be of infinite order.

<u>EXAMPLES</u> : Now let us look at a few examples:

1) **The set Z of all integers under the composition of addition**

Now let us check whether the above set forms a group.

Z = { ....-3 , -2 , -1 , 0 , 1 , 2 , 3 , 4 ,............. }

a) CLOSURE : The sum of any two integers a , b is also an integer.

-1 + 5 = 4 ; -3 + (-9) = -12 ;

In general ,

$a + b \epsilon Z \quad \forall a , b \epsilon Z$

b) **ASSOCIATIVITY :**

2 + (3 + (-8)) = - 3

(2 + 3) + (-8) = -3

In general

a + (b + c) = (a + b) + c $\forall$ **a , b, c** ϵ **Z**

c) **EXISTENCE OF IDENTITY : Let us say that an Identity element "x" exists in this set.**

Then $\forall$ a , x ϵ Z **, by definition of identity,**

a + x = x + a = a

Consider a + x = a

$\Longrightarrow$ **x = 0**

Similarly for x + a = a

$\Longrightarrow$ **x = 0**

Therefore ,the identity element for the group is 0.

d) **EXISTENCE OF INVERSE : : Let us say that for an element "a" the inverse exists in this set .**

Let it be denoted by "y".

Then , by definition ,

a + y = y + a = "identity element"

$\Longrightarrow$ **a + y = y + a = 0 (as obtained from point (c))**

$\Longrightarrow$ y= -a $\forall$ **a** ϵ **Z**

So , the inverse of a given element "a" is "-a" for this group.

2) **The set of all positive rational numbers** Q_+ **under the composition defined by a * b =** $\frac{ab}{3}$

Check whether the above set forms an abelian group.

a) **CLOSURE : If a,b** ϵ Q_+ **,then** $\frac{ab}{3}$ **is also a rational number (by definition of rational numbers)**

b) **ASSOCIATIVITY : If a , b , c** ϵ Q_+ **,then**

a * (b * c) = a * ($\frac{bc}{3}$**)** **(UNDERSTAND THE MEANING OF OPERATION "*" HERE)**

$= \frac{abc}{9}$

$(a * b) * c = (\frac{ab}{3}) * c$

$= \frac{abc}{9}$

Hence , associativity holds .

c) EXISTENCE OF IDENTITY : Let "x" be the identity ,then

$a * x = x * a = a$

$\Rightarrow a * x = \frac{ax}{3} = a$

$\Rightarrow x = 3$

d) EXISTENCE OF INVERSE : : Let "y" be the inverse ,then

$a * y = y * a = E$

$\Rightarrow a * y = E$

$\Rightarrow \frac{ay}{3} = 3$ **(Putting E =3 from point (c))**

$\Rightarrow y = \frac{9}{a}$

Therefore Q_+ forms a group under the given composition.

Now, let us check whether it is abelian.

$a * b = \frac{ab}{3}$ **(1)**

$b * a = \frac{ba}{3}$ **(2)**

Since the process of multiplication is commutative , therefore a * b = b * a .

Therefore ,the group is an abelian group.

3) **The set W = { 1 , -1 , i , - i } under the composition of multiplication of elements.**

CLOSURE : We can check for any two elements $\forall$ a , b ϵ W

$a . b = c \ \epsilon \ W, \ \forall a, b \ \epsilon \ W$

Checking for all combinations:

$1 x - i = - i \ \epsilon \ W$,

$i x i = -1 \ \epsilon \ W$..........

hence ,closure holds.

ASSOCIATIVITY : $\forall$ a , b , c ϵ W

a * (b * c) = (a * b) * c

Checking for all combinations:

-1 (i x –i) = - 1

(-1 x i) x – i = -1…………….

EXISTENCE OF IDENTITY : Let "x" be the identity element for the set W.

Then ,$\forall$ a ϵ W , by definition of identity,

a x = x a = a

$\Longrightarrow$ a x = a

$\Longrightarrow$ x =1 .

E = x =1

EXISTENCE OF INVERSE : Let "y" denote the inverse of the element "a".

Then ,$\forall$ a ϵ W , by definition of inverse,

a y = y a = E

a y =1

$\Longrightarrow$ y = (1/a) .

Note : Unlike the groups in examples 1 and 2 above , this is a group with 4 elements. So ,this is a group with ORDER 4 .

For such FINITE groups , all the group postulates can be easily checked using the GROUP MULTIPLICATION TABLE.

The GROUP MULTIPLICATION TABLE enlists all the elements of the finite group both as a row as well as a column. After going through the REARRANGEMENT THEOREM you will be able to explain that each ROW AND COLUMN OF THE GROUP MULTIPLICATION TABLE CONTAINS EACH ELEMENT ONLY ONCE.

Constructing the GROUP MULTIPLICATION TABLE.

Step 1 : Write down all "h " elements of the group in "h" rows and "h" columns .

Step 2 : In the grid/table thus created use the composition "*" to fill in the cells such that

(i th Row-element) * (j th column element) = result to be filled in the ij th cell

It is shown how a group multiplication table is constructed by using the example 3 given above. Further ,it is illustrated how the group-postulates can be very conveniently checked using such a table.

⟹All elements	(Composition: multiplication)	1	-1	I	-i
	1	1	-1	I	- i
	-1	-1	1	-i	i
	i	i	-i	-1	1
	-i	-i	i	1	-1
	⟹All Elements				

From the above table it can be clearly seen that:

CLOSURE : All elements when multiplied amongst themselves yield elements of the set W.

ASSOCIATIVITY :

a=- i , b = - i, c = -1

(-i)in 4^{th} row multiplied to (-i) in fourth col =-1

(a b) c =((- i)(- i))(-1) = -1 (-1) =1

Similarly a(bc)=1

EXISTENCE OF IDENTITY : Corresponding to each element in the group it can be seen that The first col(and the first row) gives back to us the same element , corresponding to the postulate

a E =a . (row-→ Ea = a)

Therefore E =1

EXISTENCE OF INVERSE OF EACH ELEMENT : The inverse of each element can be checked by looking at the row and finding the cell in which "E" or the identity element of the group occurs . The element in the col heading corresponds to its inverse.

For eg , we want to find the inverse of "-i".

We find that E=1 occurs in the (fourth column of the table or) third col of multiplied elements. The column heading corresponding to this 1 is "i".

So the inverse of "-i" is "i".

THE GROUP MULTIPLICATION TABLE FOR { 1,-1 , i ,- i }

(mult)	1	-1	i	-i
1	1	-1	i	- i
-1	-1	1	-i	I
i	i	-i	-1	1
-i	-i	i	1	-1

PRACTICE PROBLEMS

Determine whether the following sets are groups or not under the given law of composition . Draw the Group Multiplication Table wherever necessary. If the set happens to be a finite group, also give the order of the group.

1) **The set consisting of real numbers 1 and -1 with ordinary multiplication as the law of composition.**
2) **The set consisting of just unity under ordinary multiplication.**
3) **The set consisting of complex numbers {1 , -1 , i , -i } under the operation of multiplication. (where the complex number i = √ -1).**
4) **The set of all m X n matrices under matrix addition.**
5) **The set of all real numbers under addition.**
6) **The set of all natural numbers under addition.**
7) **The set Q of all rational numbers under multiplication.**

8) The set Q_0 of all non-zero rational numbers under multiplication.
9) The set Z of all integers with the composition defined by a * b = a + b + 1 .
10) The set Q_1 of all rational numbers excluding 1 with the operation defined by a *b = a + b – ab.
11) The set Q' of all rational numbers other than -1 with the operation defined by a* b=a + b + ab.
12) The set { 0 , 1 , 2 } with the operation defined by a * b = |a-b|.
13) The set of all n X n non-singular matrices having their elements as real numbers with respect to the composition of matrix multiplication. (Assume that matrix multiplication is associative.) (Is the above group abelian ? What if the elements of the matrices are rational numbers/complex numbers ?).
14) The set of matrices :

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

(Assume that matrix multiplication is associative.)
15) The set of matrices where $\alpha \in$ R ,under the composition of matrix multiplication:

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

(Assume that matrix multiplication is associative.)
16) The set of elements { 0 , 1 } under the composition "ADDITION MODULO 2".
(addition modulo 2 $\Longrightarrow$ add the numbers ,divide by 2. The remainder corresponds to the result. That is, the remainder is to be written as the result of the composition.)
17) Does the set of matrices G = { E , A , B } form a group wrt matrix multiplication.,where:

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Add a mimimum number of matrices to this set so that it becomes a group.
18) The set G of matrices under the composition of matrix multiplication ,where:

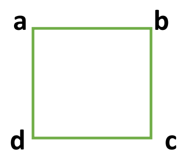
$$G = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$: where a and b are not both zero at the same time.

(Assume that matrix multiplication is associative.)
19) The set G of matrices under the composition of matrix multiplication ,where:

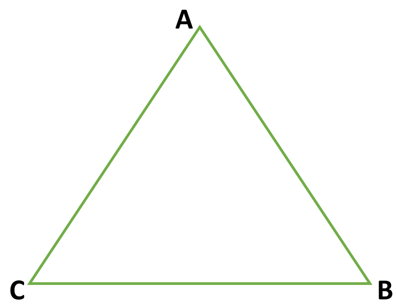
$$G = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} : a, b, c \in Z \text{ (set of integers)} \right\}$$

Determine whether this group is abelian or not.
20) Consider a square kept on the table . Imagine a situation where you were to leave the room and your friend were to change the orientation of this square such that when you come back and view the square again you are not able to determine any change in the configuration of the square. Such operations are called symmetry operations .THINK OF ALL THE SYMMETRY OPERATIONS OF THE SQUARE WHICH WILL LEAVE IT INVARIANT. Given that such a specified

set of symmetry operations forms a group , construct the GROUP MULTIPLICATION TABLE FOR ALL SYMMERTY OPERATIONS OF A SQUARE.



21) **Think of all the symmetry operations of an EQUILATERAL TRIANGLE which will leave it invariant. Given that such a specified set of symmetry operations forms a group , construct the GROUP MULTIPLICATION TABLE for all the symmetry operations of an equilateral triangle.**



Acknowledgement and Declaration :

These problems have been collected from various textbooks, internet resources and question-papers of various universities .I therefore express my sincere gratitude to all the authors without naming them specifically and individually.

ERRATA

In the file study_material_3 a couple of typographical errors occurred .The same may be corrected.

- **The angular momentum in component form is given by:**

  $\vec{L}$ = ((y p_z -z p_y) ,(z p_x – x p_z) ,(x p_y – y p_x))

- **PB [L_y , p_x] = [L_2 , p_1] = - p_3**

  Also note the negative sign in : [L_y , p_x] = - p_z