B . Sc (Hons) PHYSICS

SEMESTER-6, SECTION-B

ADVANCED MATHEMATICAL PHYSICS-II

SUBMITTED BY: SEEMA TRAMA

The Indices in Poisson Brackets

The meaning of indices:

$$[\ u\ ,\ v\ ] = \sum_{i=1}^{n} \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}$$

In the above expression the summation is over the index 'i'. After giving all possible values to 'i' from 1,2 .. n ,it will be summed out and hence it is called a "dummy index". Any "dummy index " can be replaced by any other dummy index without changing the "meaning" of the expression.

Now let us look at the following expression:

We have to prove that the Poisson Bracket <u>**[q_i , p_i] = 1**</u>

Here in [q_i , p_i] , <u>**'i' is no longer a dummy index**</u> **.**

<u>**For example**</u> **,it could mean [x , p_x] , [y , p_y] or [z , p_z] .**

<u>**So, in order not to change the meaning of the expression , we should choose another index which works as the dummy**</u>**.**

So, the Poisson Bracket [q_i , p_i] ,in expanded form would look like:

$$[\ q_i\ ,\ p_i\ ] = \sum_{j=1}^{n} \frac{\partial q_i}{\partial q_j}\frac{\partial p_i}{\partial p_j} - \frac{\partial q_i}{\partial p_j}\frac{\partial p_i}{\partial q_j} \qquad (1)$$

Here 'j' is chosen as the dummy index . We might as well have chosen l,m,λ… but not 'i' ,in this case.

The meaning of the summation:

Now , we consider [x , p_x] ,that is [q_i , p_i] taking i=1 meaning we are talking about the coordinate x.

Suppose n=3 ,then taking j=1 (corresponding to x) ,

j=2 (corresponding to y) and

j=3 (corresponding to z) , such that,

q_1 = x q_2 = y q_3 = z

p_1 = p_x p_2 = p_y p_3 = p_z

Letting the dummy index 'j' take all possible values so that it is summed out:

$$[\, x , p_x \,] = \frac{\partial x}{\partial x}\frac{\partial p_x}{\partial p_x} + \frac{\partial x \partial p_x}{\partial y \partial p_y} + \frac{\partial x}{\partial z}\frac{\partial p_x}{\partial p_z} - \left(\frac{\partial x}{\partial p_x}\frac{\partial p_x}{\partial x} + \frac{\partial x \partial p_x}{\partial p_y \partial y} + \frac{\partial x}{\partial p_z}\frac{\partial p_x}{\partial z} \right)$$

$$= 1 + 0 + 0 + 0 + 0 + 0 = 1$$

(Using the property of the Kronecker Delta Function:

δ_{ij} = 1 if i = j ,

= 0 otherwise)

Therefore , knowing that x , y , z are independent and having seen how a dummy index behaves, we can very compactly write equation (1) as

$$[\, q_i , p_i \,] = \sum_{j=1}^{n} \frac{\partial q_i}{\partial q_j}\frac{\partial p_i}{\partial p_j} - \frac{\partial q_i}{\partial p_j}\frac{\partial p_i}{\partial q_j} \qquad (1)$$

[q_i , p_i] = δ_{ij} = 1 ,

knowing that only one term in the summation will survive , the one corresponding to which

$q_i = q_j$ and $p_i = p_j$.

(Also note, the above was just an example and q_i could be any of the generalized coordinates of the system and p_i the corresponding generalized momenta. They don't necessarily have to be Cartesian coordinates .)

Now find the Poisson Brackets :

$[q_i , q_j]$

$[p_i , p_j]$

$[q_i , p_j]$

Angular Momentum and Poisson Brackets

Now let us look at the Poisson brackets involving angular momentum.

In order to differentiate it from the Lagrangian ,we denote the angular momentum by $\vec{L} = \vec{r} \times \vec{p}$.

The angular momentum has the following components in three dimensions:

$\vec{L} = ((y\, p_z - z\, p_y) ,(z\, p_y - y\, p_z) ,(x\, p_y - y\, p_x))$

Let us try to evaluate the Poisson Bracket $[L_x , p_y]$

METHOD 1 : Using properties of PBs and components of $\vec{L}$

$[L_x , p_y] = [y\, p_z - z\, p_y , p_y]$

Using the property $[u - v , w] = [u , w] - [v , w]$

$[L_x, p_y] = [y p_z, p_y] - [z p_y, p_y]$

Using the property $[ab, c] = a[b, c] + [a, c]b$

$[L_x, p_y] = (y[p_z, p_y] + [y, p_y] p_z) - (z[p_y, p_y] + [z, p_y] p_y)$

$= 0 + 1(p_z) - 0 - 0 = p_z$

Similarly we can evaluate PBs for $[L_y, p_z]$, $[L_z, p_x]$

METHOD 2 : Using tensors

Using $\vec{L} = \vec{r} \times \vec{p}$ such that in component form

$\vec{L} = ((y p_z - z p_y), (z p_y - y p_z), (x p_y - y p_x))$

and the tensorial expression for cross product : $\vec{a} \times \vec{b} = \epsilon_{ijk} a_j b_k$

$[L_i, p_j] = [\epsilon_{ijk} x_j p_k, p_j]$

where ϵ_{ijk} is the Levi-Civita (or Alternating)Tensor

$\epsilon_{ijk} = +1$ for cyclic order of indices (eg:i,j,k ; j,k,i ; k,i,j ;123 , 312 , 231)

$= -1$ for non-cyclic order of indices

$= 0$ otherwise (for any two or all three indices equal)

AND

$x_1 = x$, $x_2 = y$, $x_3 = z$; $p_1 = p_x$, $p_2 = p_y$, $p_3 = p_z$

$[L_i, p_j] = \epsilon_{ijk}[x_j p_k, p_j]$

$= \epsilon_{ijk} [x_j p_k, p_j]$

$= \epsilon_{ijk} (x_j [p_k, p_j] + [x_j, p_j] p_k)$

$= \epsilon_{ijk} (x_j [p_k , p_j] + [x_j , p_j] p_k)$

$= 0 + \epsilon_{ijk} [x_j , p_j] p_k$

$= \epsilon_{ijk} (1)p_k$

Note that this is a more general way and proves all the above three PBs and more….

For example, consider the PB $[L_y , p_x] = [L_2 , p_1] = \epsilon_{213} (1)p_3$

213 being a non-cyclic order of indices $\epsilon_{213} = -1$

Therefore the PB $[L_y , p_x] = [L_2 , p_1] = - p_3$

$[L_y , p_x] = p_z$

Now try to evaluate the PBs

$[L_x , L_y]$

$[L_y , L_x]$

and the most general $[L_i , L_j]$

<u>Those students who did not opt for the course on tensors can use method 1(if they so desire).</u>

<u>**Addendum to study_material_2**</u>

Those students who want to study the concept of "Virtual Displacement" in greater detail can go through the following readings:

1) **On virtual displacement and virtual work in Lagrangian dynamics**
   Subhankar Ray and J Shamanna
2) **2006-823: LEARNING THE VIRTUAL WORK METHOD IN STATICS: WHAT IS A COMPATIBLE VIRTUAL DISPLACEMENT?**
   Ing-Chang Jong, University of Arkansas

<u>**ERRATA**</u>

1) **In the file study_material_2.pdf , a typographic error inadvertently occurred. The same may be corrected.**

   6.9) $[\, p_z \,, L_y \,] = -\, p_x$.

2) **In the file study_material_1.pdf the fig for Q10 is:**

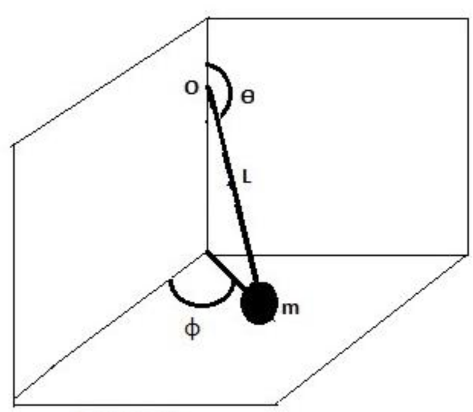

fig for Q10