B.Sc (Hons) PHYSICS

SEMESTER-6, SECTION-B

ADVANCED MATHEMATICAL PHYSICS-II

SUBMITTED BY: SEEMA TRAMA

The following are the REFERENCES for the 'CALCULUS OF VARIATIONS AND CLASSICAL DYNAMICS' section of the syllabus

H.Goldstein :Classical Mechanics**(HG)**

K. C. Gupta :Classical Mechanics of Particles and Rigid Bodies.**(KCG)**

Rana and Joag :Classical Mechanics**(RJ)**

Gupta and Kumar :Classical Mechanics **(GK)**

Patrick Hamill :A Student's Guide to Lagrangians and Hamiltonians**(PH)**

CONTINUING FROM WHERE WE LEFT OFF…..

The concept of The Principle of Least Action has been discussed in the class and the Euler Lagrange Equation (for one dependent variable as well as many dependent variables) has been derived .Many problems of Calculus of Variations have been done in the class .The concept of Lagrangian has been introduced in the class.

READING ASSIGNMENT FOR WEEK-1:

PH: Chapter-1(pg1-30, 36-38)

1.2(Generalized coordinates)

1.3(Generalized Velocity)

1.4(Constraints)

1.5(Virtual Displacements)

1.6(Virtual Work and Generalized Force)

1.9(Dynamics)

1.10(Obtaining the Equation of Motion)

1.10.1-10.1.2(EOM in Newtonian and Lagrangian Mechanics)

1.11.1(Generalized Momentum)

1.11.4(Conservation of energy and work and the introduction to the concept of the Hamiltonian.)

3.2-3.4 Lagrange Dynamics, Hamilton's Principle

NOTE : THE SYLLABUS SPECIFIC TOPICS ARE MENTIONED ABOVE. HOWEVER, IT WOULD BE IN YOUR BEST INTEREST TO READ THE WHOLE OF CHAPTER 1 OF THE READING PH.

LEARNING OBJECTIVES

After going through the above concepts you will be able to answer the following questions:

1) Write down the statement of the Hamilton's Principle. Derive the Euler-Lagrange equation for the Lagrangian using Hamilton's Principle.
2) A set of problems to determine the Lagrangian is attached for practice. You are to do only part a) and b) of the problem set first. Part c) is to be done later. (Although the syllabus mentions 'simple pendulum and one –dimensional harmonic oscillator' ,some other variations of the harmonic oscillator problem which involve two masses and hence two equations of motion and a couple of 3-d system are also given .These will help you to grasp the concept . Hints to solve these problems are also given.)
   NOTE:In the problem set you are asked to determine the Lagrangians of simple systems. The first problem : the simple pendulum ,is solved step-wise and in complete detail for your reference.
3) Define generalized coordinates and corresponding generalized velocities.
4) Using the expression for velocity derived above , write down the generalized Kinetic Energy of an N-particle system.(For this you can refer to RJ or GK)
5) What is principle of virtual work ? Why should we use it ? Are there any advantages and subsequent applications of this principle.
6) Using the concept of a virtual displacement derive the expression of generalized force.
7) Define the generalized momenta for an N particle system and find its time derivative for a conservative system.

USING THE LAGRANGE'S METHOD TO DERIVE THE EQUATION OF MOTION OF A SYSTEM:

The following steps detail how to determine the Lagrangians of simple systems.
The problem : the simple pendulum ,is solved step-wise and in complete detail for your reference.

LAGRANGIAN OF THE SIMPLE PENDULUM :

Let us look at how to find the Lagrangian of a Simple Pendulum of length l and mass m.

Step 1 : We write the x and y coordinates of the mass.(If there is more than one mass do it for each mass.)

$x = l*\sin\theta$ (Here we are approximating that the arc along which the pendulum moves is linear)

$y = l*\cos\theta$ (The y coordinate is always chosen wrt a suitable reference level.We have chosen the line passing through the x-axis as the reference level,as shown in the diagram.)

Step 2 : We write the velocity corresponding to the mass. The mass moves in the x-y plane.

Here we note that in the Cartesian System :

velocity squared : $v^2 = (\dot{x}^2 + \dot{y}^2)$

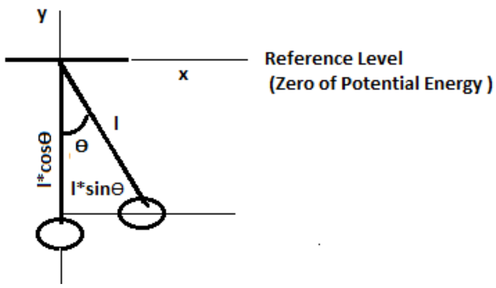
Converting the above to plane polar coordinates

$$v^2 = (\dot{r}^2 + r^2\dot{\theta}^2)$$

Since r = l =constant (This is the constraint of the system)

Therefore , the velocity squared is :

$$v^2 = (l^2\dot{\theta}^2)$$



Step 3 : We write the potential and Kinetic energy of the system.

Potential Energy = V = -m*g * y = -m*g*(l*cosΘ)

Kinetic Energy = T = $\frac{m}{2}v^2 = \frac{m}{2}(l^2\dot{\theta}^2)$

Step 4 : Lagrangian L = T - V

$$L = \frac{m}{2}(l^2\dot{\theta}^2) + m*g*(l*\cos\Theta)$$

Step 5 : **Now use the Euler –Lagrange Equation to find the Equations of Motion.**

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

Where q = the only generalized coordinate in this case = Θ

$$\frac{\partial L}{\partial \Theta} = -m\,g\,l\sin\Theta$$

$$\frac{\partial L}{\partial \dot{\Theta}} = m\,l^2\dot{\theta}$$

Therefore ,the Euler-Lagrange Equation is :

$$m\,l^2\ddot{\Theta} + m\,g\,l\sin\Theta = 0$$

s.t the Equation of motion is (for the small angle approximation sinΘ ~ Θ) :

$$\ddot{\Theta} + \frac{g}{l}\Theta = 0.$$

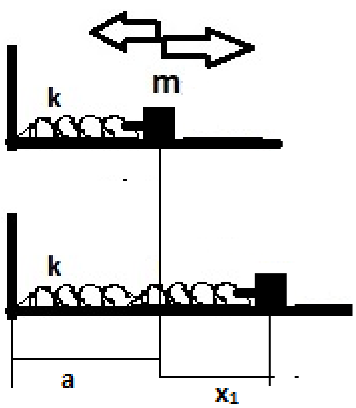
PRACTICE PROBLEMS

Assuming that each of the following systems are conservative (that is , no dissipative forces are at work),write down for the following systems:

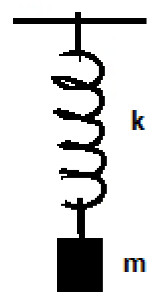
a) the Lagrangian b) the Equation(s) of Motion c)the Hamiltonian

1) A simple pendulum of mass 'm' and fixed length 'l' suspended by an inextensible string from a rigid support.

2) A spring of spring-constant 'k' connected to a wall on one side and a mass 'm' on the other, and placed on a horizontal frictionless table.(a horizontal harmonic oscillator)



Horizontal spring-mass system (Horizontal Harmonic Oscillator) fig for Q2



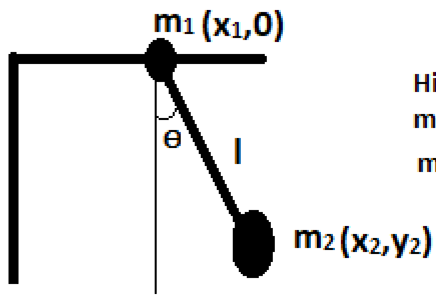
Vertical spring-mass system (Vertical Harmonic Oscillator) fig for Q3

3) A spring of spring-constant 'k' connected to an immovable rigid support on one side(top) and a mass 'm' at the bottom and suspended vertically. (a vertical harmonic oscillator)

4) A free particle in:

- **4.1) the Cartesian Coordinate System.**
- **4.2) the Cylindrical Coordinate System.**
- **4.3) the Spherical Polar Coordinate System.**

5) A pendulum has mass 'm2' at one end and mass 'm1' at the point of support. It moves in a vertical plane under gravity. Mass 'm1' can move on a horizontal line lying in the plane in which 'm2' moves .The masses are separated by an inextensible string of fixed length 'l' (See fig 1):



Hint: Write the x and y coordinates of both masses

mass1: $x = x_1$, $y = y_1 = 0$

mass2: $x = x_2 = x_1 + l*\sin\theta$

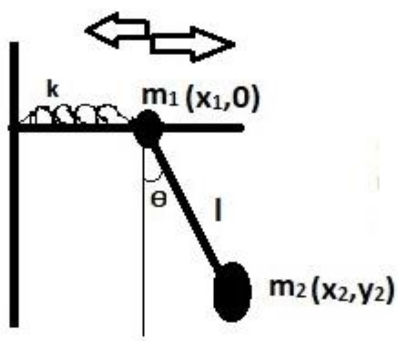
$y = y_2 = -l*\cos\theta$

Now you calculate the velocities $\dot{x}_1$, $\dot{x}_2$, $\dot{y}_1$, $\dot{y}_2$ and write the Kinetic Energy of both the masses and sum it up to find the total kintic energy of the system.

$T = \frac{1}{2} m_1 \ (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_1 \ (\dot{x}_2^2 + \ \dot{y}_2^2)$.Similarly find the total potential energy of the system.

Lagrangian : L = T – V.

6) A small bead of mass m_1 is initially at rest on a smooth horizontal wire and is attached to a point on the wire by a massless spring of spring-constant 'k' and unstretched length 'a'. A mass m_2 at the end of the wire of length 'l' is freely suspended from the bead.(see fig 2).The mass m_1 is moved to the right and released thereby setting the system into motion.



Hint:

mass m_1 : $x = x_1$

$y = y_1 = 0$

mass m_2 : $x = x_2 = x_1 + l*\sin\theta$

$y = y_2 = - l*\cos\theta$

Now find the velocities of and and then the total kinetic energy of the system.

Similarly find the total potential energy of the system.

7) Two-mass three-spring system shown below:

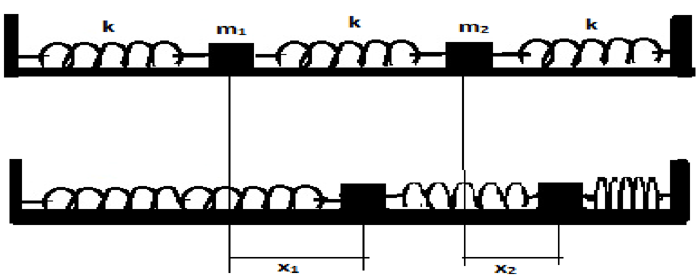
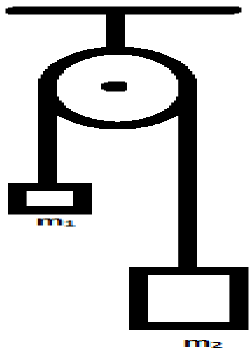


fig for Q7

8) The Atwood Machine System shown below:



The length of the string is 'a'.
fig forQ8

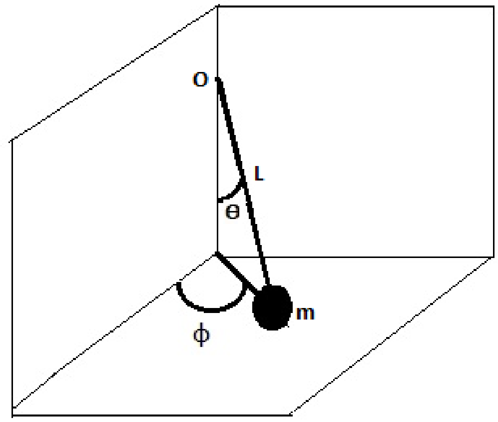


fig for Q10

9)

9.1) A particle moving under an attractive central force which obeys the inverse square law.

9.2) A particle moving under a conservative force with potential V (ρ , ϕ ,z) in cylindrical coordinates.

10) A mass m at the end of a rubber band of length 'L' is whirling about point O as shown in the fig below. The mass 'm' is executing 'free' oscillations , i.e. in all three dimensions subject to the constraints of the system.(Hint: If we replace L by 'r' ,it would be easy to determine the coordinate system best suited to this problem .Also note the 'rubber band ' means it is a variable length system.)

11) A particle is moving on a plane curve xy = c (where 'c' is a constant) under the action of gravity(y-Axis vertical).Obtain the Lagrange's Equation of Motion.

12) A particle of mass 'm' is tied to one end of a massless spring (spring constant 'k' and unstretched length r_0).The other end of the spring is fixed to a point P on a smooth horizontal plane on which the particle is moving .If the instantaneous position of the particle is (r,Θ) , then obtain the Lagrangian of this system

Acknowledgement and Declaration :

These problems have been collected from various textbooks, internet resources and question-papers of various universities .I therefore express my sincere gratitude to all the authors without naming them specifically and individually.