Course - B.Sc Physics (Hons)

Semester - IV

Paper Name - Mathematical Physics III.

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Instructions to the Students-

1) Dear Students in the last e-notes I started Lapalace transform and shared e-notes related to this topic

2) Here I am giving you some problems related to the earlier topic. Solve these problems and send it to my email. My email id is Surbhik1@gmail.com.

Problems

1.Q → Evaluate $L\left\{\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right\}$

2.Q → Evaluate $\int_0^\infty e^{-3t}\sin t\,dt$

3.Q → Evaluate $L\left\{\frac{1-\cos t}{t}\right\}$

P.T.O

3. Instructions Continued.----

I am sharing e-notes of the next topic that is Laplace transform of unit step function and Laplace transform of periodic function. and Convolution of Laplace transform and Convolution theorem of Laplace transform.

$$= \frac{s+1-1}{s\sqrt{s+1}\,(\sqrt{s+1}+1)} = \frac{1}{\sqrt{(s+1)}\left\{\sqrt{(s+1)}+1\right\}}$$

$$\therefore L\left[erf_c\,(\sqrt{t})\right] = \boxed{\frac{1}{\sqrt{s+1}\left\{\sqrt{s+1}+1\right\}}}$$

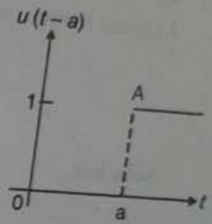
Ans.

46.17 UNIT STEP FUNCTION

With the help of unit step functions, we can find the inverse transform of functions, which cannot be determined with previous methods.

The unit step function $u(t-a)$ is defined as follows:

$$\boxed{u(t-a) = \begin{cases} 0, & \text{when } t < a \\ 1, & \text{when } t \geq a \end{cases}} \quad \text{where } a \geq 0$$



46.18 LAPLACE TRANSFORM OF UNIT FUNCTION

$$L[u(t-a)] = \frac{e^{-as}}{s}$$

Proof. $L\left[u(t-a)\right] = \int_0^\infty e^{-st} u(t-a)\,dt = \int_0^a e^{-st}.0dt + \int_a^\infty e^{-st}.1\,dt = 0 + \left[\frac{e^{-st}}{-s}\right]_a^\infty$

$$\boxed{L[u(t-a)] = \frac{e^{-as}}{s}}$$

Proved.

Example 28. *Express the following function in terms of unit step functions and find its Laplace transform:*

$$f(t) = \begin{cases} 8, & t < 2 \\ 6, & t \geq 2 \end{cases}$$

Solution. $f(t) = \begin{cases} 8+0, & t < 2 \\ 8-2, & t \geq 2 \end{cases} = 8 + \begin{cases} 0, & t < 2 \\ -2, & t \geq 2 \end{cases} = 8 + (-2)\begin{cases} 0, & t < 2 \\ 1, & t \geq 2 \end{cases} = 8 - 2u(t-2)$

$$L\{f(t)\} = 8L(1) - 2\,L\,u(t-2) = \frac{8}{s} - 2\frac{e^{-2s}}{s}$$

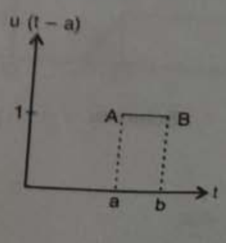
Ans.

Example 29. *Draw the graph of $u(t-a) - u(t-b)$.*

Solution. As in Art 46.17 the graph of $u(t-a)$ is a straight line parallel to t-axis from A to ∞.

Similarly, the graph of $u(t-b)$ is a straight line parallel to t-axis from B to ∞.

Hence, the graph of $u(t-a) - u(t-b)$ is AB. **Ans.**



10. $f(t)=\begin{cases}4, & 0<t<1\\ -2, & 0<t<3\\ 5, & t>3\end{cases}$ **Ans.** $\dfrac{4-6e^{-s}+7e^{-3s}}{s}$

46.21. PERIODIC FUNCTIONS

Let $f(t)$ be a periodic function with period T, then

$$\boxed{L[f(t)]=\frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}}$$

Proof. $L[f(t)]=\int_0^\infty e^{-st}f(t)\,dt=\int_0^T e^{-st}f(t)\,dt+\int_T^{2T}e^{-st}f(t)\,dt+\int_{2T}^{3T}e^{-st}f(t)\,dt+\ldots$

Substituting $t=u+T$ in second integral and $t=u+2T$ in third integral, and so on.

$$L[f(t)]=\int_0^T e^{-st}f(t)\,dt+\int_0^T e^{-s(u+T)}f(u+T)\,du+\int_0^T e^{-s(u+2T)}f(u+2T)\,du+\ldots$$

$$=\int_0^T e^{-st}f(t)\,dt+e^{-sT}\int_0^T e^{-su}f(u)\,du+e^{-2sT}\int_0^T e^{-su}f(u)\,du+\ldots$$

$$[f(u)=f(u+T)=f(u+2T)=f(u+3T)=\ldots]$$

$$=\int_0^T e^{-st}f(t)\,dt+e^{-sT}\int_0^T e^{-st}f(t)\,dt+e^{-2sT}\int_0^T e^{-st}f(t)\,dt+\ldots$$

$$=\left[1+e^{-sT}+e^{-2sT}+e^{-3sT}+\ldots\right]\int_0^T e^{-st}f(t)\,dt \qquad \left[1+a+a^2+a^3+\ldots=\frac{1}{1-a}\right]$$

$$=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt.$$

Proved.

Example 38. *Find the Laplace transform of the waveform*

$$f(t)=\left(\frac{2t}{3}\right),\ 0\le t\le 3.$$

Solution.

$$L[f(t)]=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$$

$$L\left[\frac{2t}{3}\right]=\frac{1}{1-e^{-3s}}\int_0^3 e^{-st}\left(\frac{2}{3}t\right)dt=\frac{1}{1-e^{-3s}}\,\frac{2}{3}\left[\frac{te^{-st}}{-s}-(1)\frac{e^{-st}}{s^2}\right]_0^3$$

$$=\frac{2}{3}\,\frac{1}{1-e^{-3s}}\left[\frac{3e^{-3s}}{-s}-\frac{e^{-3s}}{s^2}+\frac{1}{s^2}\right]=\frac{2}{3}\cdot\frac{1}{1-e^{-3s}}\left[\frac{3e^{-3s}}{-s}+\frac{1-e^{-3s}}{s^2}\right]$$

$$=\frac{2e^{-3s}}{-s\left(1-e^{-3s}\right)}+\frac{2}{3s^2}$$

Ans.

Example 39. *Draw the graph and find the Laplace transform of the triangular wave function of period 2 C given by*

$$f(t)=\begin{cases}t, & 0<t\le C\\ 2C-t, & C<t<2C\end{cases}$$

(Uttarakhand, II Semester, June 2007)

Solution. Period $=2C=T$

Laplace transform of periodic function $f(t)$

$$L\{f(t)\}=\frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$$

$$L\{f(t)\}=\frac{1}{1-e^{-2Cs}}\int_0^{2C}e^{-st}f(t)\,dt \qquad (T=2c)$$

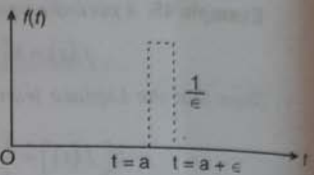
On putting the values of $f(t)$, we get

6. $f(t)=\begin{cases}\cos\omega t, & 0<t<\dfrac{\pi}{\omega}\\[2mm] 0, & \dfrac{\pi}{\omega}<t<\dfrac{2\pi}{\omega}\end{cases}$ **Ans.** $\dfrac{s}{\left(s^2+\omega^2\right)\left(1-e^{-\frac{\pi s}{\omega}}\right)}$

7. $f(t)=\begin{cases}t, & 0<t<1\\ 0, & 1<t<2\end{cases}\quad f(t+2)=f(t)$ **Ans.** $\dfrac{1-e^{-s}(s+1)}{s^2\left(1-e^{-2s}\right)}$

8. $f(t)=\begin{cases}\dfrac{2t}{T}, & 0\le t\le\dfrac{T}{2}\\[2mm] \dfrac{2}{T}(T-t), & \dfrac{T}{2}\le t\le T\end{cases}\quad f(t+T)=f(t)$ **Ans.** $\dfrac{2}{Ts^2}\tanh\dfrac{sT}{4}-\dfrac{1}{s\left(e^{\frac{sT}{2}}+1\right)}$

46.22 CONVOLUTION THEOREM



If $\quad L\left[f_1(t)\right]=F_1(s)$ and $L\left[f_2(t)\right]=F_2(s)$

then $\quad L\left\{\int_0^t f_1(x)f_2(t-x)\,dx\right\}=F_1(s)\cdot F_2(s)$

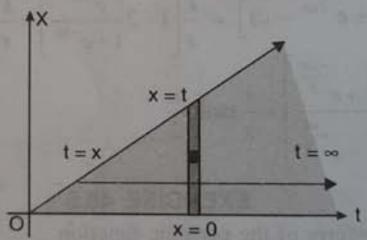
or $\quad \boxed{L^{-1}\left(F_1(s).F_2(s)\right)=\int_0^t f_1(x)f_2(t-x)\,dx}$

Proof. We have

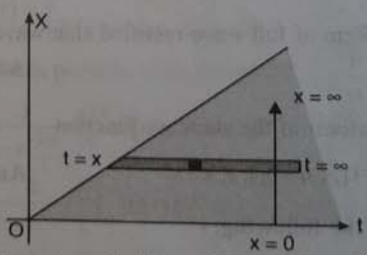
$$L\left\{\int_0^\infty f_1(x)f_2(t-x)\,dx\right\}=\int_0^\infty e^{-st}\left[\int_0^t f_1(x)f_2(t-x)\,dx\right]dt \qquad \text{(By Definition)}$$

where the double integral is taken over the infinite region in the first quadrant lying between the lines $x = 0$ and $x = t$.

Here first we are integrating w.r.t. "x", within limits $x = 0$ and $x = t$, and then we will integrate w.r.t. "t" with limits $t = 0$ and $t = \infty$.



On changing the order of integration first we integrate w.r.t. "t" with limits $t = x$ and $t = \infty$ and then w.r.t. "x" with limits $x = 0$ and $x = \infty$.



On changing the order of integration, the integral becomes

$$\int_0^\infty dx\left[\int_x^\infty e^{-st} f_1(x).f_2(t-x)\,dt\right]$$

$$=\int_0^\infty dx\left[\int_x^\infty e^{-s(t-x+x)} f_1.(x).f_2(t-x)\,dt\right]=\int_0^\infty dx\left[\int_x^\infty e^{-s(t-x)}.e^{-sx} f_1(x).f_2(t-x)\,dt\right]$$

$$= \int_0^\infty e^{-sx} f_1(x)\,dx \left[\int_x^\infty e^{-s(t-x)} f_2(t-x)\,dt\right] = \int_0^\infty e^{-sx} f_1(x)\,dx \left[\int_x^\infty e^{-sz} f_2(z)\,dz\right]$$

$$= \int_0^\infty e^{-sx} f_1(x)\,dx \int_0^\infty e^{-sz} f_2(z)\,dz,$$

[Put $t - x = z \Rightarrow dt = dz$]
Lower limit $x - x = z \Rightarrow z = 0$]

$$= \int_0^\infty e^{-sx} f_1(x) F_2(s)\,dx = \left[\int_0^\infty e^{-sx} f_1(x)\,dx\right] F_2(s) = F_1(s)F_2(s)$$ **Proved.**

Example 46. *Find the Laplace transform of* $\int_0^t e^x . \sin(t-x)\,dx$

Solution. By Convolution Theorem

$$L \int_0^t f_1(x)\, f_2(t-x)\,dx = F_1(s) . F_2(s)$$

$$\Rightarrow L \int_0^t e^x . \sin(t-x)\,dx = L(e^x) . L \sin t = \frac{1}{s-1}\,\frac{1}{s^2+1} = \frac{1}{(s-1)(s^2+1)}$$ **Ans.**

Note. Convolution Theorem is generally used to find Inverse Laplace transform of the product of two functions, discussed in the next chapter.

46.23 LAPLACE TRANSFORM OF BESSEL FUNCTIONS $J_0(x)$ and $J_1(x)$

We know that

$$J_n(x) = \frac{x^2}{2^n\,\Gamma(n+1)}\left[1 - \frac{x^2}{2.(2n+2)} + \frac{x^4}{2.4\,(2n+2)(2n+4)} - \dots\right]$$

$$J_0(t) = \left[1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \frac{t^6}{2^2.4^2.6^2} + \dots\right]$$

Taking Laplace transforms of both sides, we have

$$LJ_0(t) = \frac{1}{s} - \frac{1}{2^2}.\frac{2!}{s^3} + \frac{1}{2^2.4^2}\frac{4!}{s^5} - \frac{1}{2^2.4^2.6^2}\frac{6!}{s^7} + \dots$$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1.3}{2.4}\left(\frac{1}{s^4}\right) - \frac{1.3.5}{2.4.6}\left(\frac{1}{s^6}\right) + \dots\right]$$

$$= \frac{1}{s}\left[1 + \left(-\frac{1}{2}\right)\left(\frac{1}{s^2}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{1}{s^2}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{1}{s^2}\right)^3 + \dots\right]$$

(By Binomial theorem)

$$= \frac{1}{s}\left[1 + \frac{1}{s^2}\right]^{-\frac{1}{2}}$$

$$= \frac{1}{s}\left[\frac{s^2+1}{s^2}\right]^{-\frac{1}{2}} = \frac{1}{s}\left[\frac{s^2}{s^2+1}\right]^{\frac{1}{2}} = \frac{1}{\sqrt{s^2+1}}$$ ... (1) **Ans.**

We know that $Lf(at) = \frac{1}{a} F\left(\frac{s}{a}\right)$

$$LJ_0(at) = \frac{1}{a}\frac{1}{\sqrt{\frac{s^2}{a^2}+1}} = \frac{1}{\sqrt{s^2+a^2}}$$ [From (1)]

$$L J_1(x) = -L J_0'(x) = -[s\,LJ_0(x) - J_0(0)] = -\left[s.\frac{1}{\sqrt{s^2+1}} - 1\right] = 1 - \frac{s}{\sqrt{s^2+1}}$$ **Ans.**

Laplace inverse transform

Start

If $f(t)$ is given $(t>0)$

Then Laplace transform is defined as

$$L\{f(t)\} = \bar{f}(s) = \int_0^{\infty} e^{-st} f(t)\,dt$$

Lapace inverse

$$L^{-1}\{\bar{f}(s)\} = f(t)$$

$L\{t\} = \frac{1}{s^2}$ $\qquad L^{-1}\{\frac{1}{s^2}\} = t$

$L(1) = \frac{1}{s}$ $\qquad L^{-1}\{\frac{1}{s}\} = 1$

$L\{t^3\} = \frac{\lfloor 3}{s^{3+1}} = \frac{6}{s^4}$ $\qquad L^{-1}\{\frac{1}{s^4}\} = \frac{t^3}{6}$

① $L\{t^n\} = \frac{\lfloor n+1}{s^{n+1}}$ $\qquad L^{-1}\{\frac{1}{s^{n+1}}\} = \frac{t^n}{\lfloor n+1}$

② $L\{e^{at}\} = \frac{1}{s-a}$ $\qquad L^{-1}\{\frac{1}{s-a}\} = e^{at}$

3) $L\{\sin at\} = \frac{a}{s^2+a^2} \Rightarrow L^{-1}\{\frac{1}{s^2+a^2}\} = \frac{1}{a}\sin at$

④ $L\{\cos at\} = \frac{s}{s^2+a^2} \Rightarrow L^{-1}\{\frac{s}{s^2+a^2}\} = \cos at$

⑤ $L\{\sinh at\} = \frac{a}{s^2-a^2} \Rightarrow L^{-1}\{\frac{1}{s^2-a^2}\} = \frac{1}{a}\sinh at$

⑥ $L\{\cosh at\} = \frac{s}{s^2-a^2} \Rightarrow L^{-1}\{\frac{s}{s^2-a^2}\} = \cosh at$

⑦ $L\{1\} = \frac{1}{s}$ $\qquad \Rightarrow L^{-1}\{\frac{1}{s}\} = 1 \quad (n=0)$

⑧ $L\{t\} = \frac{1}{s^2}$ $\qquad \Rightarrow L^{-1}\{\frac{1}{s^2}\} = t$

Inverse Laplace Transform

If $F(s)$ is the Laplace tranfform of a function $f(t)$, then $f(t)$ is known as inverse Laplace tranform.

$$\text{If } L\{f(t)\} = F(s)$$

$$\text{then } L^{-1}[F(s)] = f(t)$$

where L^{-1} is called the inverse Laplace tranform operator.

Important formulae:-

1. $L^{-1}\left(\frac{1}{s}\right) = 1$
2. $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{\lfloor n-1}$.
3. $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$
4. $L^{-1}\frac{s}{s^2-a^2} = \cosh at$
5. $L^{-1}\frac{1}{s^2-a^2} = \frac{1}{a}\sinh at$
6. $L^{-1}\frac{1}{s^2+a^2} = \frac{1}{a}\sin at$
7. $L^{-1}\frac{s}{s^2+a^2} = \cos at$
8. $L^{-1}F(s-a) = e^{at}f(t)$
9. $L^{-1}(1) = \delta(t)$
10. $L^{-1}\left(\frac{1}{t}F(t)\right\} = \int_0^t f(t)\,dt$

(3a)

Laplace transform properties.

If $L\{f(t)\} = \bar{f}(s)$ then

1) $L\{t f(t)\} = -\frac{d}{ds}\bar{f}(s)$

Laplace inverse properties

If $L^{-1}\{\bar{f}(s)\} = f(t)$ then

1) $L^{-1}\{\frac{d}{ds}\bar{f}(s)\} = -t L^{-1}\{\bar{f}(s)\}$
$= -t f(t)$

① (multiply by t property → changed to → laplace inverse of derivative)

② $L\{\frac{f(t)}{t}\} = \int_s^{\infty} \bar{f}(s)\,ds$ ② $L^{-1}\{\int_s^{\infty} \bar{f}(s)\,ds\} = \frac{f(t)}{t}$

② Divide by 't' property changed to Lapalace transform of integral.

③ $L\{\frac{d}{dt}f(t)\} = s\bar{f}(s) - \bar{f}(0)$ ③ $L^{-1}\{s\bar{f}(s)\} = \frac{d}{dt}f(t)$ if $f(0) = 0$

L^{-1}(constant) = no inverse

③ Derivative property changed to multiply by 't' property

④ $L\{\int_0^t f(t)\,dt\} = \frac{\bar{f}(s)}{s}$ ④ $L^{-1}\{\frac{\bar{f}(s)}{s}\} = \int_0^t f(t)\,dt$

④ Integral property changed to divide by 't' property

⑤ $L\{e^{at} f(t)\} = \bar{f}(s-a)$ ⑤ $L^{-1}\{\bar{f}(s-a)\} = e^{at} f(t)$

⑤ Integral property ch

⑥ Partial fraction method.