Course - B.Sc Physics (Hons)

Semester - IV

Paper Name - Mathematical Physics III

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Instructions to the Students. →

1) Dear Students I am continuing with my previous notes. I am starting with a new topic that is Convolution and Convolution theorem. Convolution theorem is a derivation type topic. It comes in examination as a long question type.

2) The second topic is the solution of wave eq^n by using fourier transform. In the last semester you have already solve the one dimensional wave eq^n by using separation of variable method. This topic is also a long answer type question and some of boundary conditions may be change in the exam.

Convolution

Convolution is an operator which works on two signals.

Time domain → frequency domain

in time domain if multiplication occurs → In frequency domain it is convolution denoted as as star symbol but it is not the multiplication sign. ie, $*$

if in time domain it is convolution ie, $*$ $\xrightarrow{\text{then}}$ In frequency domain it is multiplication.

Property of Convolution → If $M(t)$ is a function then

$$\text{if } M(t) * \delta(t-t_0) = M(t-t_0) \rightarrow \text{Shifted function}$$

This means that when a function convolutes with a shifted impulse ie, $\delta_1(t=t_0)$ at results in a shifted function.

$$\frac{dx}{dt} = 5x+3y$$

$$\frac{dy}{dt} = x+7y. \quad \text{initial conditions are } x(0)=5 \text{ and } y(0)=1$$

Convolution Theorem for fourier transform.

Let Convolution of $f(x)$ and $g(x)$ is defined in the interval $(-\infty,\infty)$ as $f*g = \int_{-\infty}^{\infty} f(u)\,g(x-u) = h(x)$

Convolution theorem for fourier transform – The fourier transform of convolution of $f(x)$ and $g(x)$ is the product of their fourier transform i.e)

$$F\{f*g\} = F\{f(x)\}\cdot F\{g(x)\}$$

Proof:– $F\{f(x)\} = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx. \quad \text{——— (1)}$

$$\Rightarrow F\{f*g\} = \int_{-\infty}^{\infty} e^{isx}\left[\int_{-\infty}^{\infty} f(u)\,g(x-u)\,du\right]dx.$$

→ first integ w.r.t. u then w.r.t. x.

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{isx} f(u)\,g(x-u)\,du\,dx.$$

Changing the order of integration,

$$= \int_{-\infty}^{\infty} f(u)\cdot\left[\int_{-\infty}^{\infty} e^{isx}\, g(x-u)\,dx\right] du.$$

put, $x-u=t \Rightarrow dx = dt$

$$= \int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{+\infty} e^{is(u+t)}\, g(t)\,dt\right] du$$

$$= \int_{-\infty}^{\infty} f(u) \cdot e^{isu} \left[\int_{-\infty}^{\infty} e^{ist} g(t) dt \right] du$$

$$= \int_{-\infty}^{+\infty} f(x) e^{isx} dx \int_{-\infty}^{\infty} e^{isx} g(x) dx$$

Remark

Replaced u and t by x

using (definite integral property by changing variable do not change the value of integral)

$$= F\{f(x)\} \cdot F\{g(x)\}$$

$$\left[\therefore F\{f * g\} = F\{f(x)\} \times F\{g(x)\} \right]$$

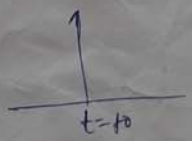
Convolution → is an operator (± mathematical operator) which works on two signal

time domain → frequency

multiplication occurs in time domain → X then it converted to → $*$ - Convolution

$*$ convolution in freq → X - multiplication

Property → $M(t)$ → is a function.



$M(t) * \delta(t - t_0) \rightarrow = M(t - t_0)$ → why modulation performed.

↳ A function convolves with a shifted impulse (at $t = t_0$) results in a shifted function. that is the $M(t - t_0)$ takes a place of impulse.

Solution of wave Equation → Application of fourier Transform

Use the method of fourier transform to determine the displacement $y(x,t)$ of an infinite string, given that the string is initially at rest and the initial displacement is $f(x)$, $-\infty < x < \infty$. Show that the Sol^n can also be put in the form

$$y(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] \quad \text{--- (i)}$$

Sol^n: → Displacement $y(x,t)$ of any point of an infinite string is governed by the wave eq^n

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{--- (ii)}$$

where $-\infty < x < \infty$, $t > 0$

with Conditions (i) $y(x,0) = f(x)$.

(i) $y(x,0) = f(x)$

(ii) $\frac{\partial y}{\partial t} = 0$ at $t = 0$ (Since the string is initially at rest)

From (ii),

$$F\left\{\frac{\partial^2 y}{\partial t^2}\right\} = c^2 F\left\{\frac{\partial^2 y}{\partial x^2}\right\}$$

or, $$\int_{-\infty}^{\infty} \frac{\partial^2 y}{\partial t^2} e^{-ipx}\, dx = c^2 \int_{-\infty}^{\infty} \frac{\partial^2 y}{\partial x^2} e^{-ipx}\, dx. \;?$$

By fourier transform of a derivative

or, $$\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty} y\, e^{-ipx}\, dx = c^2 (ip)^2 \bar{Y}(p,t)$$

or, $$\frac{d^2 \bar{y}}{dt^2} + c^2 p^2 \bar{y} = 0 \quad \text{--- (iii)}$$

Its Solution is,

$$\bar{Y} = A\cos cpt + B\sin cpt \quad \text{—— (iv)}$$

But $\frac{\partial y}{\partial t} = 0$ at $t = 0$, and so $\frac{d\bar{y}}{dt} = 0$ at $t = 0$

from (iv), $\quad 0 = \frac{d\bar{y}}{dt} = cp\left[-A\sin cpt + B\cos cpt\right]_{t=0}$

$$0 = cp\left[A\cdot 0 + B\right] \text{ or } B = 0 \quad \text{—— (v)}$$

$\therefore$ from (iv), $\bar{y} = A\cos cpt$.

$\because F\{y(x,0)\} = F\{f(x)\}$

$$\Rightarrow \bar{Y}(p,0) = \int_{-\infty}^{\infty} f(x)\, e^{-ipx}\, dx = \bar{f}(p)$$

$$\Rightarrow \bar{Y}(p,0) = \bar{f}(p)$$

$$\Rightarrow A = \bar{f}(p) \quad \text{—— (va)}$$

Now (iv) becomes, putting the value of A in (iv)

$$\bar{Y} = \bar{f}(p)\cos cpt \quad \text{—— (vi)}$$

Taking inverse fourier transform of (vi) we get

$$y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(p)\cos cpt \cdot e^{ipx}\, dp$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(p)\cdot\left(\frac{e^{icpt} + e^{-icpt}}{2}\right) e^{ipx}\, dp$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(p)\left[\frac{e^{ip(x+ct)} + e^{ip(x-ct)}}{2}\right] dp$$

$$= \frac{1}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(p)\left[e^{ip(x+ct)} + e^{ip(x-ct)}\right] dp\right]$$

$$= \frac{1}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(p)\, e^{ip(x+ct)}\, dp + \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(p)\, e^{ip(x-ct)}\, dp\right] \quad \text{—— (vii)}$$

using the fact that $g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{g}(p)\, e^{ipx}\, dp$ we get

$$y(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right].$$

$\bar{Y} = A\cos cpt$

$\therefore \bar{Y}(p,0) = \bar{f}(p)$

$\bar{f}(p) = A$