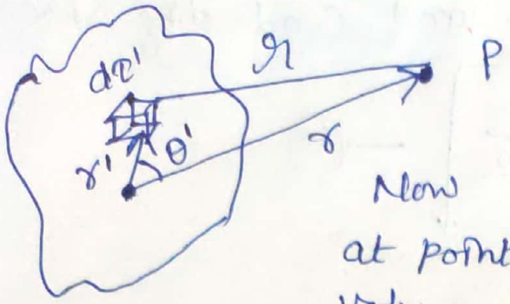
MULTIPOLE EXPANSION



If the total charge distributed uniformly. So $\rho(r')$ is vol. charge density.

Now we can write Potential at point P. and Consider a small volume $d\tau'$. Now Potential due to a small vol. $d\tau'$ is.

$$\Rightarrow \boxed{V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\mathfrak{r}}\,\rho(r')\,d\tau'} \quad (1)$$

In diagram $r' \Rightarrow$ position vector in the volume

$r \Rightarrow$ Position vector out side the volume.

$\mathfrak{r} = |r - r'|$ $\qquad \rho(r') =$ Vol. charge density depends upon Prime vector.

$$\Rightarrow \mathfrak{r}^2 = r^2 + r'^2 - 2rr'\cos\theta'$$

$$\Rightarrow \mathfrak{r}^2 = r^2\left[1 + \frac{r'^2}{r^2} - 2\left(\frac{r'}{r}\right)\cos\theta'\right]$$

Here $\epsilon = \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta'$

$$\Rightarrow \mathfrak{r}^2 = r^2(1+\epsilon) \Rightarrow \mathfrak{r} = r\sqrt{1+\epsilon}$$

$$\Rightarrow \mathfrak{r} = r(1+\epsilon)^{1/2}$$

By binomial Theorem.

$$\Rightarrow \frac{1}{\mathfrak{r}} = \frac{1}{r}(1+\epsilon)^{-1/2}$$

$$\Rightarrow \frac{1}{\mathfrak{r}} = \frac{1}{r}\left(1 - \frac{\epsilon}{2} + \frac{3}{8}\epsilon^2 - \cdots\right)$$

Here ϵ is very less than one. So by binomial Theorem and higher term will be skipped

$$\Rightarrow \boxed{\frac{1}{\mathfrak{r}} = \frac{1}{r} - \frac{\epsilon}{2r} + \frac{3}{8r}\epsilon^2} \quad (2)$$

Put the value of ϵ in Eq. (2)

$$\Rightarrow \frac{1}{\mathfrak{r}} = \frac{1}{r} - \frac{1}{2r}\left[\frac{r'^2}{r^2} - \frac{2r'}{r}\cos\theta'\right] + \frac{3}{8r}\left[\frac{r'^2}{r^2} - \frac{2r'}{r}\cos\theta'\right]^2$$

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$$\Rightarrow \frac{1}{\mathscr{r}} = \frac{1}{r} + \frac{r'\cos\theta'}{r^2} + \frac{1}{r^3}\left[\frac{3}{2}r'^2\cos^2\theta' - \frac{r'^2}{2}\right] \quad (5)$$

$$+ \; - - - \quad (3)$$

Put the value of $\frac{1}{\mathscr{r}}$ in Eq^n (1)

So Potential due to Total volume at point P.

$$\Rightarrow V(r) = \frac{1}{4\pi\varepsilon_0}\left[\int \frac{\rho(r')\,d\tau'}{r} + \int \frac{\rho(r')\,r'\cos\theta'\,d\tau'}{r^2} + \int \frac{\rho(r')}{r^3}\left(\frac{3}{2}r'^2\cos^2\theta' - \frac{r'^2}{2}\right)d\tau' + - - - - \right] \quad (4)$$

Eq^n (4) is the desired result – the MultiPole Expansion of V in powers of $\left(\frac{1}{r}\right)^n$.

First term	$(n=1)$	Monopole Contribution
IInd term	$(n=2)$	Dipole Contribution
IIIrd term	$(n=3)$	Quadrupole Contribution
4th term	$(n=4)$	Octopole Contribution

and so on

From eq^n (4)

$$V_{monopole} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')\,d\tau'}{r}$$

$$V_{dipole} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')\,r'\cos\theta'\,d\tau'}{r^2} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')\,\vec{r}'\cdot\hat{r}}{r^2}\,d\tau'$$

So. $$\boxed{V_{dip}(r) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}} \quad (6)$$

$$\boxed{V_{dip}(r) = \frac{\hat{r}}{4\pi\varepsilon_0 r^2}\int r'\rho(r')\,d\tau'} \quad (7)$$

and

$$V_{qud}(r) = \frac{1}{4\pi\varepsilon_0 r^3}\int \rho(r')\left[\frac{3}{2}(r'\cos\theta')^2 - \frac{(r')^2}{2}\right]d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0 r^3}\int \rho(r')\left[\frac{3}{2}(r'\cdot\hat{r})^2 - \frac{r'^2}{2}\right]d\tau'$$

Here the integral part is quadrupole moment of the distribution.

$$\Rightarrow Q = \frac{1}{2}\int \rho(r')\left[3(r'\cdot\hat{r})^2 - (r')^2\right]d\tau'$$

or

$$\Rightarrow Q = \frac{1}{2}\int \rho(r')(3z^2 - r'^2)\,d\tau'$$

or

$$\boxed{Q = \frac{1}{2}\int \rho(r')(r')^2(3\cos^2\theta' - 1)\,d\tau'} \quad (8)$$

* Question :- (1) Example 3.10 (Griffiths)

(2) Problem :(3.33)
(3) Problem (3.43)] — Griffiths

(4) Problem (3.45)

(5) How would you define the octopole moment? Express the octopole term in the multipole Expansion in terms of the octopole moment.