Applications of transforms to boundary value problems

Determine the distribution of temperature in the semi-infinite medium $x \geq 0$ when the end $x = 0$ is maintained at zero temperature and the initial distribution of temperature is $f(x)$.

Let $u(x,t)$ be the temperature at any point x and at any time t. Solving the heat flow equation.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (x > 0, t > 0) \qquad \text{--- (1)}$$

$$u(x,0) = f(x) \qquad \text{--- (2)}$$

$$u(0,t) = 0 \qquad \text{--- (3)}$$

Taking the fourier sine transform of eq (1) we get

Denoting $F_s[u(x,t)] = \overline{u_s}$

$\because \dfrac{d\,\overline{u_s}}{dt} = c^2 F_s\left[\dfrac{\partial^2 u}{\partial x^2}\right]$

$= c^2 \Big[\ \cancel{s\,u(\ )}_{x \to 0}$

$= c^2 \left[s\,u(0,t) - s^2 \overline{u_s} \right]$

or $\dfrac{d\overline{u_s}}{dt} + c^2 s^2 \overline{u_s} = 0$ —— (4)

the fourier sine transform of eq (2) is,

$\overline{u_s} = \overline{f_s}(s)$ at $t = 0$ —— (5)

$\because$ Solving (4) & (5)

$\dfrac{d\overline{u_s}}{dt} + c^2 s^2 \overline{f_s}(s)$

$\because \overline{u_s} = \overline{f_s}(s)\, e^{-c^2 s^2 t}$ —— (6)